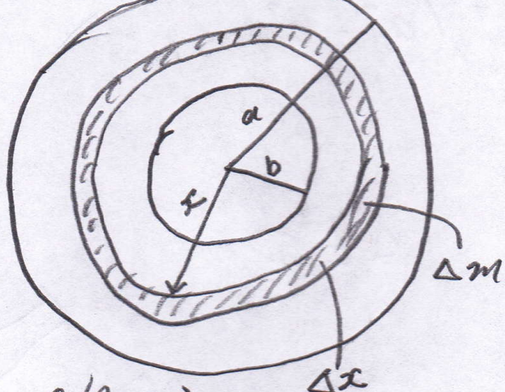
2010 Q8.

(a) See proof 2013 Q8(a)

(b)(i)



$\rho = \frac{\text{mass}}{\text{Area}} \Rightarrow M = \rho(\text{Area})$

$= \frac{M}{\pi a^2 - \pi b^2}$

$= \frac{M}{\pi(a^2-b^2)}$

$M = \rho(\text{Area})$

$\Rightarrow dM = \rho\, dA$

$= \frac{M}{\pi(a^2-b^2)} \cdot 2\pi x\, dx$

$\boxed{\begin{aligned} \text{Area} &= \pi x^2 \\ \frac{dA}{dx} &= 2\pi x \\ dA &= 2\pi x\, dx \end{aligned}}$

$= \frac{2M}{a^2-b^2} \cdot x\, dx$

$dI = x^2\, dm$ (Definition)

$dI = \frac{2M}{a^2-b^2} x^3 \cdot dx$

$\int dI = \int_b^a \frac{2M}{a^2-b^2} x^3\, dx$

$\Rightarrow \quad I = \left[\frac{2M}{a^2-b^2} \cdot \frac{x^4}{4}\right]_b^a$

$= \frac{2M}{(a^2-b^2)4}(a^4-b^4)$

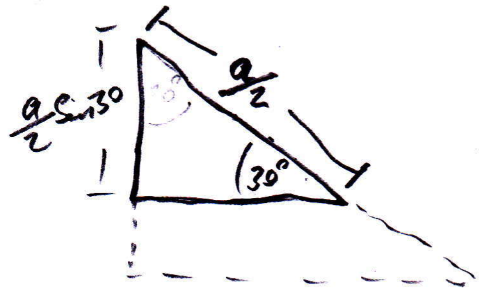
Since: $a^4 - b^4 = (a^2)^2 - (b^2)^2$

$= (a^2+b^2)(a^2-b^2)$

We have:

$$I = \frac{2M}{4(a^2-b^2)} \cdot (a^2+b^2)(a^2-b^2)$$

$$= \frac{M(a^2+b^2)}{2} \quad QED$$



(ii) Gain in KE = Loss in PE

$$\frac{1}{2} I w^2 + \frac{1}{2} MV^2 = Mgh.$$

$$\Rightarrow \frac{1}{2} I w^2 + \frac{1}{2} M(a w)^2 = Mg\left(\frac{a}{2}\sin 30\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{M(a^2+b^2)}{2}\right)w^2 + \frac{1}{2}M(aw)^2 = M\frac{a}{4}g \quad \left[\sin 30 = \frac{1}{2}\right]$$

$$\Rightarrow \frac{M(a^2+b^2)}{4} w^2 + \frac{1}{2} Ma^2w^2 = M\frac{a}{4}g$$

$$\Rightarrow w^2\left(\frac{M(a^2+b^2)}{4} + \frac{1}{2}Ma^2\right) = M\frac{a}{4}g$$

$$\Rightarrow w^2\left(\frac{a^2+b^2}{4} + \frac{a^2}{2}\right) = \frac{ag}{4}$$

$$\Rightarrow w^2\left(\frac{3a^2+b^2}{4}\right) = \frac{ag}{4}$$

$$\Rightarrow w^2 = \frac{ag}{3a^2+b^2}$$

$$\Rightarrow w = \sqrt{\frac{ag}{3a^2+b^2}}$$